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The Asteroid Identification Problem

III. Proposing Identifications

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A large fraction of asteroids have been lost shortly after discovery, thus the asteroid catalogs contain a large number of low accuracy orbits. Two of these inaccurate orbits can belong to the same physical object; the challenge is to find effective algorithms for identification. We give a new method to propose identifications of orbits, applicable in the case where each of the two observed arcs provides enough information to solve for all the orbital elements by a least-squares fit to the observations.

Even if the optimum fit solution is unique, there is a confidence region in the space of orbital elements containing orbital solutions compatible with the observations: the identification of orbits is the search for an orbital solution in the intersection of the two confidence regions. In the linear approximation there is a rigorous and simple algorithm to find the optimum joint solution and the increase in the RMS of the residuals relative to the two separate solutions. The linear approximation may fail if two poorly determined orbits are too far apart in the orbital elements' space. In this case, the linear algorithm becomes more stable when restricted to only some of the orbital elements.

Our procedure proposes orbit identification using a cascade of tests, all based upon identification metrics taking into account the difference in the orbits weighted with the uncertainty. The first test is based only upon the orbital plane; the couples of orbits compatible according to the first test are submitted to further tests using identification metrics based upon 5 and 6 orbital elements. Finally, the couples passing all tests are submitted to an accurate computation, by differential correction, of the orbit fitting both sets of observations. This procedure has been tested on a set of 100 already known identifications and was found to be effective in 99% of the cases. Finally we show that these methods have been used to obtain 152 previously unknown orbit identifications.

Key Words: **asteroids; orbits; celestial mechanics.**

1. INTRODUCTION

A large fraction of the asteroids observed so far are to be considered lost: they cannot be recovered by pointing the telescope at the predicted position only. The catalogs of asteroid orbits are therefore polluted by large numbers of low accuracy orbits that can not be easily improved by observation. Two of these inaccurate orbits can belong to the same physical object, and indeed identifications are regularly found when the orbital elements are close; an unknown number of real identifications have not been found in the existing catalogs because the orbital elements as computed are far apart. This situation requires some action to be taken because the number of low quality orbits is indeed increasing with time, notwithstanding the efforts of many dedicated people in the maintenance of catalogs, and it results in a significant waste of the precious resources of telescope and observer time.

The completion of the task requires an improvement in both the observation capability and the theoretical understanding of the problem. To contribute to the solution from the theoretical side, this paper belongs to a series dedicated to the asteroid identification problem, begun with Milani (1999), hereafter referred to as Paper I.

In general, the asteroid identification problem deals with separate sets of astrometric observations, which are assumed to form two observed arcs, each one belonging to one object; what is not known is whether the two arcs refer to the same object. The

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problem can take very different forms, depending upon the amount and distribution in time of the available observations for each arc and upon the time interval between the two arcs; a rigorous terminology to distinguish the different cases was introduced in Paper I. This paper deals with one specific case, the *orbit identification* problem: it occurs when each one of the two arcs has been observed well enough to allow for the computation of a *complete orbit*, by which we mean an orbit with all six orbital elements solved for by a least-squares fit to the observations. It needs to be stressed that most asteroid identifications performed by observers and orbit computation centers do not belong to this case, but to either the *attribution* or to the *linkage* ones. This is because a large fraction of the available observations belong to very short arcs, lasting only very few nights, and even a single night, and in these cases complete orbits either do not exist or are essentially undetermined, in a sense that is better explained in Section 3.1. Even a few orbit identifications are, however, important achievements: each one of them not only removes two low quality orbits from the catalogs but also provides a good one, which can in turn be useful in many ways, including the attribution of other short arcs, the prediction of close approaches (Milani and Valsecchi 1999), and the recovery.

Two facts are the main sources of difficulty in finding a rigorous and effective algorithm for orbit identifications. First, the number of observed asteroids is very large (and increasing fast). Even taking into account only the complete orbits there are far too many couples to subject each one to rigorous testing. In Section 5 we have used a catalog of more than 35,000 orbits for unnumbered objects, so the algorithm to *propose identifications* should test on the order of 6×10^8 couples. Despite the power of modern computers, this implies that the computations performed on each couple need to be very simple. All the algorithms we propose here involve the computation of some distances $d_i(N_1, N_2)$ between two orbits identified by the identifiers N_1, N_2, and then the couples satisfying the simple condition $d_i(N_1, N_2) < \varepsilon_i$ are selected for further investigation. These selection criteria can be applied as a cascade of filters, each selecting fewer and fewer couples until a small number of strong proposals are submitted to the final test of the least-squares fit of the orbit to the observations of both arcs.

The second fact generating difficulties is that even a complete orbit, i.e., the solution of a least-squares fit to all the observations of a given arc, does not indicate the only possible orbit for the observed object; there is around each solution a *confidence region* where the true orbit could be without significantly increasing the size of the observation residuals. Thus it is not enough to compute some distance, defined by a suitable metric, between the nominal least-squares solutions; the distance between two orbits must be computed taking into account the uncertainty of both orbits. As an example, if a well-determined orbit is inside the confidence region of a poorly determined orbit, this couple is a strong candidate for orbit identification, independent of the size of the difference between the two nominal orbits.

In this paper we propose a rigorous and computationally efficient algorithm to propose identification by means of distances between couples of orbits taking into account the *linear* approximation of the uncertainty of both orbits, as expressed by their *normal and covariance matrices* and by their geometric counterparts, the *confidence ellipsoids*. This method has been tested both on already known identifications, which have been reproduced, and in searching for new ones, of which a significant number has been found.

This paper is organized as follows. In Section 2 we introduce the idea of a penalty associated with the identification, taking into account the linearized orbit uncertainty. In Section 3 we discuss the difficulties, and necessary cautions, arising from the weakly determined orbits, from nonlinear propagation of the uncertainty, and from poor knowledge of the error statistics. In Section 4 we propose a sequence of filtering stages, based upon some different identification penalties and test them on a sample of already identified orbits. In Section 5 we describe results obtained by applying these tests to an actual asteroid orbit catalog that we have computed based on the recently available global dataset of asteroid observations. Finally, in Section 6 we draw some conclusions about the value of our methods to actually find new identifications, but also draw some conclusions on the need for further work.

2. IDENTIFICATION PENALTIES

The purpose of this section is to define a cost or penalty associated with an orbit identification. This penalty takes into account the uncertainty of both orbit determinations, but the uncertainty is not easy to define as it depends upon the timing, quantity, and quality of the astrometric observations. There are, however, rigorous and quantitative ways to describe the uncertainty of orbits based upon the normal/covariance matrix formalism, provided the orbits have been computed as a solution of a least-squares fit. In the next subsection we recall the notation and terminology used to describe a least-squares solution. In the following subsections we describe the linear identification algorithm in the unrestricted and in the restricted form, respectively.

2.1. Differential Corrections as an Optimization Algorithm

The *principle of least squares* assumes that a *target function* Q must be minimized to find the nominal solution. The target function $Q = \xi \cdot \xi / m$ is formed with the sum of squares of the residuals ξ, with $\xi \in \Re^m$. The residuals are normalized, as discussed in Section 3.3, thus Q is dimensionless. In our case, $m = 2 \cdot N_{\mathrm{obs}}$ for N_{obs} astrometric observations of two angular coordinates. The *residuals* are functions $\xi(X)$ of the *estimated parameters* $X \in \Re^N$. In the simplest problems of orbit determination $N = 6$ and X is some vector representing the orbital elements at some initial epoch t_0: in this paper $X = (a, h, k, p, q, \lambda)$ are the equinoctial elements as defined in Paper I, Section 4.1. Some of the coordinates of the vector X, e.g., in our case the mean longitude λ, are not real numbers, but are sometimes angles (defined mod(2π)), and this introduces some complications that are noted later.

Thus the target function also depends upon X, and the minimum of $Q(X)$ is obtained by solving the nonlinear equations

$$\frac{\partial Q}{\partial X} = \frac{2}{m}\xi^T B = \underline{0}; \quad B = \frac{\partial \xi}{\partial X}(X).$$

Now let the map between X and ξ be linearized in a neighborhood of some point X^*,

$$\xi = B\Delta X; \quad \Delta X = X - X^*,$$

where the target function is approximated by a quadratic form

$$Q(X) = \frac{1}{m}\Delta X^T B^T B \Delta X.$$

The equation to be solved for the minimum is the normal equation

$$B^T B \Delta X = -B^T \xi,$$

with *normal matrix* $C = B^T B$ and solution

$$\Delta X = -\Gamma B^T \xi$$

computed with the *covariance matrix* $\Gamma = C^{-1}$, which exists whenever C is positive-definite, which is generically the case for $m \geq N$. We shall of course assume that the linearization is performed around the solution X^* such that $B^T \xi = \underline{0}$; in the standard *differential corrections* procedure X^* is obtained by iterating the solution of the normal equation until convergence (pseudo-Newton method). For a standard reference on differential correction, see Cappellari *et al.* (1976).

Please note that to apply a single iteration of differential correction, and even any fixed number of iterations, is not enough to guarantee convergence; an iterative scheme with a tight convergence control needs to be used. As an example, in our programs the convergence is controlled by requiring that the *correction norm*

$$\|\Delta X\| = \sqrt{\Delta X \cdot C \Delta X / N} < \epsilon \tag{1}$$

be below a small control value ϵ to stop the iterative procedure; $\epsilon = 10^{-5}$ is used in most cases.

2.2. Linear Orbit Identification

By *orbit identification problem* we mean to find an algorithm to determine which couples of orbits, among many included in some catalog, might belong to the same object. We assume that both orbits, for which the possibility of identification is being investigated, have been obtained as solutions of a least-squares problem. Note that this is not always the case for orbit catalogs containing asteroids observed only over a short arc. There are therefore two uniquely defined vectors of elements, X_1 and X_2, and the normal and covariance matrices $C_1, C_2, \Gamma_1, \Gamma_2$ computed after convergence of the iterative differential correction procedure, that is, at X_1, X_2. The two target functions of the two separate orbit determination processes are

$$\begin{aligned} Q_i(X) &= \frac{1}{m_i}\xi_i \cdot \xi_i \\ &= Q_i(X_i) + \frac{2}{m_i}(X - X_i)\cdot C_i(X - X_i) + \cdots \\ &= Q_i^* + \Delta Q_i; \quad i = 1, 2, \end{aligned}$$

where ξ_i are the two vectors of dimensions m_i of residuals of the separate orbit determination processes.

For the two orbits to represent the same object, observed at different times, we need to find a low enough minimum for the joint target function, formed with the sum of squares of the $m = m_1 + m_2$ residuals

$$\begin{aligned} Q &= \frac{1}{m}(\xi_1 \cdot \xi_1 + \xi_2 \cdot \xi_2) = \frac{1}{m}(m_1 Q_1 + m_2 Q_2) \\ &= \frac{1}{m}(m_1 Q_1^* + m_2 Q_2^*) + \frac{1}{m}(m_1 \Delta Q_1 + m_2 \Delta Q_2) \\ &= Q^* + \Delta Q, \end{aligned}$$

where Q^* is the value corresponding to the sum (with suitable weighting) of the two separate minima, and the *penalty* ΔQ measures the increase in the target function that results from the need to use the same orbit for both sets of observations. Note that the penalty is related by a simple factor to the quantity χ^2 which is widely in use in the statistical interpretation of Gaussian distributions; we are not using the χ^2 notation to stress that our methods are independent from the statistical interpretation and indeed do not depend upon the detailed statistics of the observation errors.

The linear algorithm to solve the problem is obtained when the quasilinear approximation can be used not only locally, in the neighborhood of the two separate solutions X_1 and X_2, but even globally for the joint solution. This is a very strong assumption, because in general we can not assume that the two separate solutions are near to each other, but if the assumption is true, we can use the quadratic approximation for both penalties ΔQ_i and obtain an explicit formula for the solution of the identification problem

$$\begin{aligned} \frac{m}{2}\Delta Q(X) &\simeq (X - X_1)\cdot C_1(X - X_1) + (X - X_2)\cdot C_2(X - X_2) \\ &= X \cdot (C_1 + C_2)X - 2X \cdot (C_1 X_1 + C_2 X_2) \\ &\quad + X_1 \cdot C_1 X_1 + X_2 \cdot C_2 X_2. \end{aligned}$$

Neglecting higher order terms, the minimum of the penalty ΔQ can be found by minimizing the nonhomogeneous quadratic form of the formula above. If the new joint minimum is X_0, then by expanding around X_0 we have

$$\frac{m}{2}\Delta Q \simeq (X - X_0)\cdot C_0(X - X_0) + K$$

and by comparing the last two formulas we find

$$C_0 = C_1 + C_2$$
$$C_0 X_0 = C_1 X_1 + C_2 X_2$$
$$K = X_1 \cdot C_1 X_1 + X_2 \cdot C_2 X_2 - X_0 \cdot C_0 X_0.$$

If the matrix C_0, which is the sum of the two separate normal matrices C_1 and C_2, is positive-definite, then it is invertible and we can solve for the new minimum point

$$X_0 = C_0^{-1}(C_1 X_1 + C_2 X_2).$$

This equation has a very simple interpretation in terms of the differential correction process: at convergence in each one of the two separate pseudo-Newton iterations $X \to X_i$ with $C_i = C_i(X_i)$ and $D_i = D_i(X_i) = C_i \Delta X_i = \underline{0}$; therefore,

$$C_1(X - X_1) = D_1 = \underline{0} \quad \text{and} \quad C_2(X - X_2) = D_2 = \underline{0}$$
$$\Rightarrow (C_1 + C_2)X = C_1 X_1 + C_2 X_2.$$

The assumption that the quasilinear approach is applicable to the identification means that C_1, C_2 can be kept constant; thus they have the same value at X_1, X_2 and at X_0. Under these conditions X_0 can be interpreted as the result of the first differential correction iteration for the joint problem.

The computation of the *minimum identification penalty* $2K/m = \Delta Q(X_0)$ can be simplified by taking into account that K is translation invariant:

$$X_0 \to X_0 + V; \quad X_1 \to X_1 + V; \quad X_2 \to X_2 + V$$
$$K \to K + 2V \cdot (C_1 X_1 + C_2 X_2 - C_0 X_0) = K.$$

Then we can compute K after a translation by $-X_1$, that is, assuming $X_1 \to \underline{0}$, $X_2 \to X_2 - X_1 = \Delta X$, and $X_0 \to C_0^{-1} C_2 \Delta X$,

$$K = \Delta X \cdot C_2 \Delta X - X_0 \cdot C_0 X_0 = \Delta X \cdot \left(C_2 - C_2 C_0^{-1} C_2\right) \Delta X,$$

and by defining

$$C = C_2 - C_2 C_0^{-1} C_2,$$

we have a simple expression of K as a quadratic form:

$$K = \Delta X \cdot C \Delta X.$$

Alternatively, translating by $-X_2$, that is, with $X_2 \to \underline{0}$, $X_1 \to -\Delta X$, and $X_0 \to C_0^{-1} C_1(-\Delta X)$,

$$K = \Delta X \cdot C_1 \Delta X - X_0 \cdot C_0 X_0 = \Delta X \cdot \left(C_1 - C_1 C_0^{-1} C_1\right) \Delta X,$$

and the same matrix C can be defined by the alternative expression

$$C = C_1 - C_1 C_0^{-1} C_1; \quad K = \Delta X \cdot C \Delta X.$$

Note that both these formulas only assume that C_0^{-1} exists. Under this hypothesis

$$C = C_2 - C_2 C_0^{-1} C_2 = C_1 - C_1 C_0^{-1} C_1. \tag{2}$$

This is true in exact arithmetic, but might be difficult to realize in a numerical computation if the matrix C_0 is badly conditioned.

We can summarize the conclusions by the formula

$$Q(X) \simeq Q^* + \frac{2}{m} \Delta X \cdot C \Delta X + \frac{2}{m}(X - X_0) \cdot C_0 (X - X_0),$$

which gives the minimum identification penalty $\Delta Q(X_0) = 2K/m$ and also allows one to assess the uncertainty of the identified solution by defining a confidence ellipsoid with matrix C_0.

2.3. Restricted Orbit Identification

For the reasons discussed in Section 3, it is not always possible to use the linear identification theory based upon all six orbital elements. The question is how to use the same algorithm on a subset of the orbital elements? The answer is implicit in the arguments presented in Paper I, Section 2.3, Case 2, which we are going to use without repeating the formal proofs.

Let us suppose that the vector of estimated parameters is split into two components, along linear subspaces of the parameter space

$$X = \begin{pmatrix} L \\ E \end{pmatrix},$$

where E are elements of interest. The normal and covariance matrices C and Γ are decomposed as follows:

$$C = \begin{pmatrix} C_L & C_{LE} \\ C_{EL} & C_E \end{pmatrix}; \quad \Gamma = \begin{pmatrix} \Gamma_L & \Gamma_{LE} \\ \Gamma_{EL} & \Gamma_E \end{pmatrix}.$$

Then the uncertainty of E for arbitrary L can be described by the penalty, with respect to the minimum point E^*

$$\Delta Q \simeq \frac{2}{m}(E - E^*) \cdot C^E (E - E^*); \quad C^E = C_E - C_{EL} C_L^{-1} C_{LE}$$

and by the *marginal covariance matrix* $\Gamma_E = (C^E)^{-1}$.

Note that the *marginal normal matrix* C^E is not C_E and that to obtain the penalty of the above formula as a function of E the value of L has to be changed with respect to the nominal solution L^* of the unrestricted problem, by an amount that is a function of E:

$$L - L^* = -C_L^{-1} C_{LE}(E - E^*).$$

Let us apply this restricted penalty formula to the restricted identification problem. Let (L_1, E_1) and (L_2, E_2) be the nominal solutions for the two arcs considered separately and C_1^E and C_2^E

be the corresponding marginal normal matrices. The variables L are given as function of E by

$$\begin{cases} L_1(E) = L_1 - \left(C_L^1\right)^{-1} C_{LE}^1 (E - E_1) \\ L_2(E) = L_2 - \left(C_L^2\right)^{-1} C_{LE}^2 (E - E_2) \end{cases}. \tag{3}$$

By the same formalism of the previous subsection,

$$\frac{m}{2} \Delta Q \simeq (E - E_0) \cdot C_0^E (E - E_0) + K_E,$$

with

$$\begin{aligned} C_0^E &= C_1^E + C_2^E \\ E_0 &= \left(C_0^E\right)^{-1} \left(C_1^E E_1 + C_2^E E_2\right) \\ K_E &= \Delta E \cdot C^E \, \Delta E \\ C^E &= C_2^E - C_2^E \left(C_0^E\right)^{-1} C_2^E = C_1^E - C_1^E \left(C_0^E\right)^{-1} C_1^E. \end{aligned}$$

Note that K_E is not the same as the complete minimum penalty K of the previous section. The estimate K_E of the minimum penalty is obtained by assuming that $L = L_1(E)$ in the computation of ΔQ_1 while $L = L_2(E)$ in the computation of ΔQ_2. Thus there is, in general, no complete solution with a single $X = (L, E)$ to be fit to the observations of both arcs with penalty K_E, but such value is obtained by using $(L_1(E_0), E_0)$ in the first arc and $(L_2(E_0), E_0)$ in the second arc, E_0 being the proposed restricted identification.

We claim that $K_E \leq K$: K_E is the minimum of the penalty over the space of variables (E, L_1, L_2), while K is the minimum of the same penalty over the same space but with the additional constraint $L_1 = L_2$, and the minimum of a function can only increase when constraints are added.

In conclusion, the proposed restricted identification E_0 is not a complete identification, and the corresponding minimum penalty K_E is not the full penalty to be paid to achieve a full identification. This procedure is, however, a good way to filter the possible identifications because $K_E \leq K$: if a couple can be discarded as a possible identification with the restricted computation, because K_E is too large, then it does not need to be tested with the complete algorithm.

Note that it is also possible to define a *constrained identification* algorithm, based upon the conditional covariance matrices and the algorithm for constrained optimization on linear subspaces, as outlined in Paper I, Section 2.3.

3. PROBLEMS WITH THE LINEAR THEORY

The linear theory for orbit identification provides the most rigorous mathematical setting to solve this problem. A mathematical theorem, however, is not at all a rigorous tool unless all the hypotheses are applicable to the concrete problem to be solved. The hypotheses needed to apply the formalism of Section 2 are the following:

1. The normal matrices C_1, C_2, and $C_0 = C_1 + C_2$ are invertible and positive-definite.
2. The map between the space of the residuals ξ and the space of estimated parameters X is in the linear regime; that is, the linearized map $\Delta X = -\Gamma\, B^T \xi$ is a good approximation in a region including both orbits.
3. The observation errors are of a size consistent with the residual normalization adopted.

In this Section we discuss the applicability of these three hypotheses to the problem of orbit identification.

3.1. *Bad Conditioning*

To test and tune our algorithms, even before having access to the full dataset of observations, we have used an orbit catalog provided by Lowell Observatory containing the normal matrices C for 37,569 asteroid orbits. Each C had been computed by fitting an orbit to the observations at some *central epoch* t_0 near the center of the observation time span (E. Bowell, private communication, 1998; for a public domain version: `ftp://ftp.lowell.edu/pub/elgb`).

A positive-definite matrix has all the eigenvalues positive. This definition is computationally meaningful only provided the *conditioning number* of the matrix (the ratio of the largest to the smallest eigenvalue) is not larger than the inverse of the *machine accuracy* or relative rounding off error (typically $\simeq 10^{-16}$).

To assess how severe the conditioning problem is we have computed all the eigenvalues of all the normal matrices of the Lowell catalog and found 37,544 positive-definite matrices. In 25 cases, however, there is at least one negative eigenvalue (in 2 of these cases there are two negative eigenvalues).

Figure 1 shows the logarithm (in base 10) of the observed arc (days) versus the logarithm of the conditioning number λ_6/λ_1; only the asteroids observed for less than 1 year are shown. The matrices with conditioning number close to, or even greater than, 10^{16} are "numerically singular" and can provide information only if handled with a very stable algorithm. For the 25 cases with negative eigenvalues, the ratios $\lambda_6/|\lambda_1|$ between the largest positive and the largest absolute value of a negative eigenvalue are marked with crosses in Fig. 1. The negative eigenvalues appear only in matrices that are anyway poorly conditioned, to the point that even the matrix multiplication $B^T B$ results in large rounding off errors. The concentration of the badly conditioned and even nonpositive-definite matrices in the cases where the observed arc is less than a month is apparent.

The covariance matrices provided by the Lowell Observatory are entirely suitable for the purposes for which they are used, that is, differential correction and linearized ephemeris uncertainty. Indeed, for their purposes the less that linearity applies, the less important the accuracy of the covariance. Conversely, for our problem it is imperative that the worst cases have highly accurate

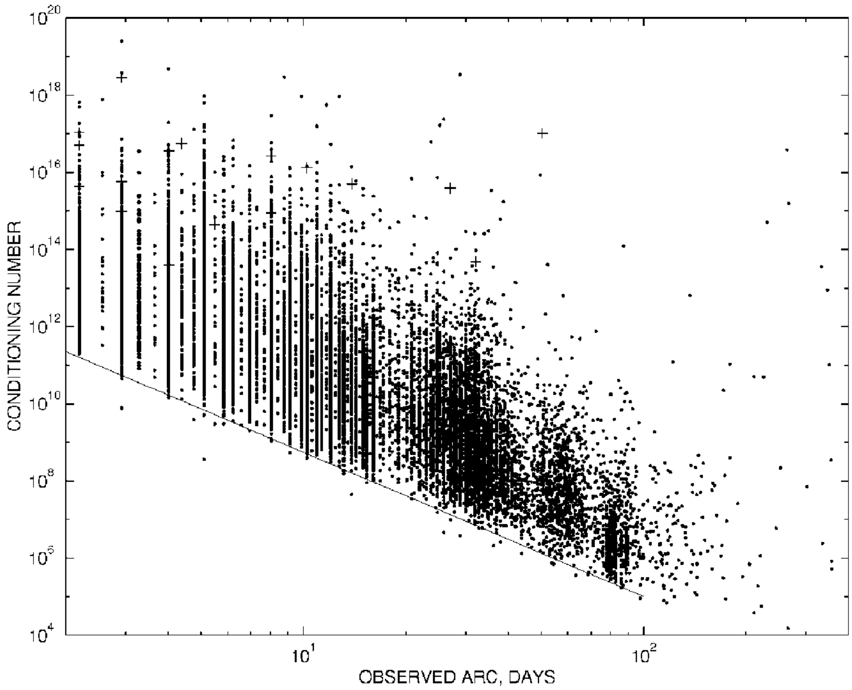

FIG. 1. Length of observed arc (days) versus conditioning number of the normal matrix from the Lowell Observatory data set. The few cases with negative eigenvalues are marked with a cross. The line plotted has a slope −3.7 in this log–log plot, indicating a strong dependence of the conditioning number upon the arc length.

uncertainty predictions in order to extend the linear theory as far as possible.

The number of badly conditioned matrices for the elements determined at the central epoch is large, but there are still enough matrices for which inversion is not a problem, provided a stable numerical method is used. We use the Cholesky algorithm, which allows one to perform the inversion of a matrix with conditioning number w by means of the inversion of an auxiliary matrix with conditioning number $\simeq\sqrt{w}$; the failures of this inversion are indeed rare.

The bad conditioning problem is becoming much worse if the normal and covariance matrices are propagated to a different epoch t (see Paper I, Section 2.4). The propagation equations contain the state transition matrices: if X_0, C_0, Γ_0 are at epoch t_0 and X, C_t, Γ_t are at epoch t, then

$$C_t = \left(\frac{\partial X_0}{\partial X}\right)^T C_0 \left(\frac{\partial X_0}{\partial X}\right); \quad \Gamma_t = C_t^{-1} = \left(\frac{\partial X}{\partial X_0}\right)\Gamma_0\left(\frac{\partial X}{\partial X_0}\right)^T. \tag{4}$$

From a two-body approximation (see Paper I, Section 4.1) it is apparent that, even without close approaches, the conditioning number goes to infinity as $t \to +\infty$ in a quadratic way. Thus the inversion $\Gamma = C^{-1}$ should always be performed only for the central epoch, when the conditioning is as good as it can get; it is numerical suicide to wait until the conditioning number gets worse to perform an inversion.

The normal and covariance matrices must be propagated to a common epoch if two orbits are to be compared for a possible identification; the best strategy is to propagate both of them separately by the matrix multiplications of the above formulas. Since this requires one to solve both the equations of motion and the variational equation accurately, it is a computationally intensive task; one may wonder if it is possible to use a simplified two-body propagation. The answer to this question is definitely negative; this can be shown by comparing the eigenvalues of the matrices propagated with different approximations.

All these cautions have been used in the computations for the tests described in this paper and are incorporated in our free software. As a result, in the test of Section 4 there are no cases in which the identification algorithm fails because of noninvertible and/or nonpositive-definite matrices.

3.2. *Nonlinearity*

If the matrices C_1, C_2, and $C_0 = C_1 + C_2$ are positive-definite it is possible to compute a linear proposed identification, but is the linear formalism a reasonable approximation of the full nonlinear problem?

This difficulty is not a separate one from the bad conditioning one: if the conditioning number of C is very large, there is a very large eigenvalue λ_6 and a very small one λ_1. Then the inverse matrix Γ must have a small eigenvalue $1/\lambda_6$ and a large eigenvalue $1/\lambda_1 = \sigma_1^2$ and a corresponding eigenspace along which the RMS uncertainty is σ_1; this is also the length of the longest semiaxis of the confidence ellipsoid defined by the inequality

$$(X - X^*) \cdot C(X - X^*) \leq 1.$$

When the longest axis of the confidence ellipsoid is very long at the central epoch the linear approximation is not adequate. However, even if it is adequate at the central epoch, the along track uncertainty grows with time, and when the confidence region spans several degrees in mean longitude the linear approximation may fail.

The problem of nonlinearity does not have an easy solution, applicable to the orbit identification problem; "brute force" (that is computationally intensive) solutions can be applied to analyze a couple of orbits for an already suspected identification, but cannot be used to propose identifications, that is, to select among hundred of millions of couples of orbits.

One possibility is to use a reduced 5×5 covariance matrix, as in our tests in Section 4. Another solution is to discard the couples of orbits X_1 and X_2 such that the difference $\Delta X = X_2 - X_1$ is too large, by using a simple metric $\|\Delta X\|$, even when the linear identification algorithm would indicate the couple as a possible identification. This solution is quite unsatisfactory, because one of the goals of the algorithm we are proposing is to allow identification of couples even with large $\|\Delta X\|$; the cases with small $\|\Delta X\|$ could be found, and indeed are at present found, with much simpler algorithms, such as the one used in Sansaturio *et al.* (1996).

An additional problem of nonlinearity is introduced if a singular set of elements is used. For example, if the least-squares fit is performed with the state vector X expressed in the usual keplerian elements $(a, e, I, \Omega, \omega, M)$, the linear approximation breaks down at the $e = 0$ and/or $I = 0$ singularities of the keplerian elements. If the confidence region contains $e = 0$ and/or $I = 0$, then the linear approximation necessarily fails within the confidence region, even when the latter is small. This problem is easily solved by using nonsingular elements such as the equinoctial ones, for which the covariance propagation is more regular; see Paper I, Section 4.1.

3.3. Normalization/Weighting of the Residuals

According to the classical probabilistic interpretation of the least-squares algorithm (Gauss 1809), the matrices Γ are the covariance matrices of a Gaussian distribution in the X space. This is based upon the assumption that linearity applies and that the observation errors have independent, unity variance normal distributions. As discussed in Paper I, Section 2.2, some unit has to be chosen to express the residuals as dimensionless numbers; the units should be chosen in such a way that the expected errors in the observational procedure are of the order of unity. The variance does not, however, need to be exactly 1; its value can be determined after the least-squares fit by a classical formula, and the matrices can be accordingly rescaled.

The problem is that the observation errors result mostly from a combination of systematic effects rather than from a random noise, which could follow a Gaussian distribution. The errors from the same observatory are typically correlated; their size depends upon the observing technology and thus changes with the observatory and over the years. Anyway, the estimation of the rescaling factor is subject to great uncertainties. In the limit case, when there are only three observations, $m = 6 = N$ and no a posteriori information is available on the observation errors. For orbits determined from a small number of observations, and with small residuals, some reasonable a priori weighting (e.g., with residuals measured in arc seconds) is more realistic than a Gaussian rescaling based upon an unrealistic error model.

These real difficulties in the use of the simplest form of the Gaussian theory lead some authors to give up whatsoever quantitative assessment of the uncertainty. This does not necessarily follow from the difficulty of using one specific, and oversimplified, error model. The information contained in the normal and covariance matrices can be used, provided we adopt a formulation independent from the Gaussian error distribution hypotheses, such as the optimization approach of Section 2. The normal matrix has an intrinsic meaning, being proportional to the Hessian matrix of the second partial derivatives of the target function, which in turn measures the overall size of the residuals. If we are prepared to decide whether a given proposed identification is either accepted or rejected on the basis of the RMS of the post-fit residuals of the observations belonging to both arcs, then we are entitled to use a prediction of the value of the postfit target function as a selection criterion.

Indeed the final acceptance of an identification requires a finer examination of the postfit residuals, possibly including outlier rejection, which is of course incompatible with the hypotheses that all the residuals follow a normal distribution. Whether it is possible to perform outlier rejection, both in single arc solutions and in proposed identification solutions, in an entirely automatic way, without intervention of an experienced human, is an open question, which we plan to address in a forthcoming paper. For the purpose of proposing identifications, however, the removal of outliers is a refinement that is meaningful only after good candidate couples are found.

Thus the algorithm for proposing identifications needs to rely only on a rough estimate of the acceptable range of values for the observation residuals. As is shown in the histograms of Section 4, a factor 2 in the controls does not matter too much, the nonlinear effects being a potential source of much larger discrepancy between the linearly proposed identification and the actual identified orbit.

4. PROPOSED ALGORITHMS AND TEST

We have selected a number of algorithms to propose identifications. They are based on the computation of a number of "distances" in the elements space X, which depend not only upon the difference $\Delta X = X_2 - X_1$ but also upon the uncertainties, as expressed by the normal matrices C_1 and C_2. We use:

1. The penalty function $d_6 = \frac{m}{2} \Delta Q_6$ for the linear identification of all the six equinoctial orbital elements.
2. The penalty function $d_5 = \frac{m}{2} \Delta Q_5$ obtained by applying the same formulas of Section 2 to a set of only five equinoctial elements, excluding the mean longitude. This computation should be much less affected by high conditioning numbers and by strong nonlinearities.
3. The penalty function $d_2 = \frac{m}{2} \Delta Q_2$ obtained by applying the same formulas of Section 2 to a set of only two elements: $p = \tan \frac{I}{2} \sin \Omega$ and $q = \tan \frac{I}{2} \cos \Omega$. This distance can be used as a preliminary filter to select the candidate couples of orbits to be subjected to more complicated computations.

The only caution to be used is to compute, for a reduced identification of the subset E of the orbital elements X, not the restriction C_E of the normal matrix but the marginal normal matrix $C^E = \Gamma_E^{-1}$ obtained by inverting the corresponding portion of the covariance matrix, as discussed in Section 2.3.

The algorithm to propose identifications should be as follows. First, over all the candidate couples, the distance d_2 should be computed and tested against some control value $d_2 < \varepsilon_2$. The logic is that the orbit plane is, in most cases, significantly better determined than the other orbital elements. Thus the corresponding marginal normal matrices are typically well conditioned, and the problems of numerical instability and nonlinearity are much less severe.

On the other hand, the use of this criterion is not enough to discriminate the strong identification candidates. If the control d_2 is kept too low, some real identifications could be missed; if it is too high, the number of proposed identifications would be huge.

Thus the following step in the algorithm is to compute, for the couples selected on the basis of the d_2 criterion, the d_5 distance and to test it against another control $d_5 < \varepsilon_5$. If this test is also positive, the d_6 distance is computed and tested $d_6 < \varepsilon_6$.

The problem is to determine values of the controls ε_2, ε_5, and ε_6 in such a way that real identifications are not discarded and the number of proposed identifications is not too large. These "distances" are dimensionless, and they do not have a straightforward physical interpretation; thus it is not possible to decide a range of acceptable values of these controls a priori in such a way that they should be satisfied by all real identifications. These acceptable ranges would have to accommodate all the possible effects of the three difficulties discussed in Section 3, namely, bad conditioning, nonlinearity, and problems in residual weighting.

Therefore we have adopted a criterion based upon experimental evidence: we have computed the values of d_2, d_5, and d_6 found in a test set of already identified orbits, as discussed in the following subsection.

4.1. *Test on 100 Previously Known Cases*

To test the proposed algorithm and also to tune the control values of the different distances to be used to propose identifications, we have selected a set of already identified orbits with the following criteria:

1. The asteroids have been observed at exactly two oppositions.
2. For each opposition, there is an arc of at least 6 days with at least five observations.
3. The semimajor axis is less than 6 AU (no trans-Neptunians, for which different criteria should be used).

We have used a set of 100 examples, satisfying the above conditions, for which observational data are available from the Minor Planet Center Extended Computer Service. For all, we have recomputed the best-fit orbit and the normal/covariance matrices for each one of the two arcs containing observations from a single opposition.

These nominal solutions, and the corresponding matrices, are computed for the epoch of the last observation in each arc. Then we have propagated all the orbital elements, and all the normal and covariance matrices, to a common epoch (we have used $t_0 = 2{,}447{,}000.5$ JD; for the dependence of the results upon this date, see Section 5). This propagation was done with accurate numerical solutions of the N-body equations of motion and of their variational equations, while the normal and covariance matrices were both propagated by means of equations (4), with no further inversions after the single one performed at the observations epoch.

With all the state vectors and normal/covariance matrices referred to the same epoch, the algorithms of Section 2 can be applied, and we have used them according to the same procedure outlined at the beginning of this Section, namely, by computing d_2, d_5, and d_6 for each test couple, formed by two observed arcs known to be of the same asteroid.

The results of the computation of our three metrics are shown in the histograms of Figs. 2–4. In Fig. 2 we show the results for the inclination-only distance d_2; it is apparent that this distance has always a low value, $\leq$10 in 89% of the test cases, $\leq$30 in 97%, with a maximum value of 80.4. This is, however, a loose criterion that can be used only as a preliminary filter, since in a large catalog with tens of thousands of orbits it would be satisfied by millions of couples.

The distance d_5 being small is obviously a much stronger constraint on the couples of orbits to be identified. Indeed it has a small value for many of the actually identified orbits: $\leq$10 in 59% of the test cases and $\leq$30 in 73%. It has, however, as shown in Fig. 3, a moderate value in a number of cases: $30 \leq d_5 \leq 1{,}000$ in 18% of the cases, and an even larger value in 9%.

The distance d_6 is the one associated with the fully linear identification algorithm. Its value is always larger than that of d_5 (this is just an example of the property $K_E \leq K$ proven in Section 2.3). Nevertheless, the value of d_6 is low in most cases: $\leq$30 in 70% of our test sample; a low to moderate value $\leq$1000 covers 86% of the cases, as shown in Fig. 4.

A final test is based upon Eq. (2), proposing two different formulas to compute the matrix C, therefore K, therefore the

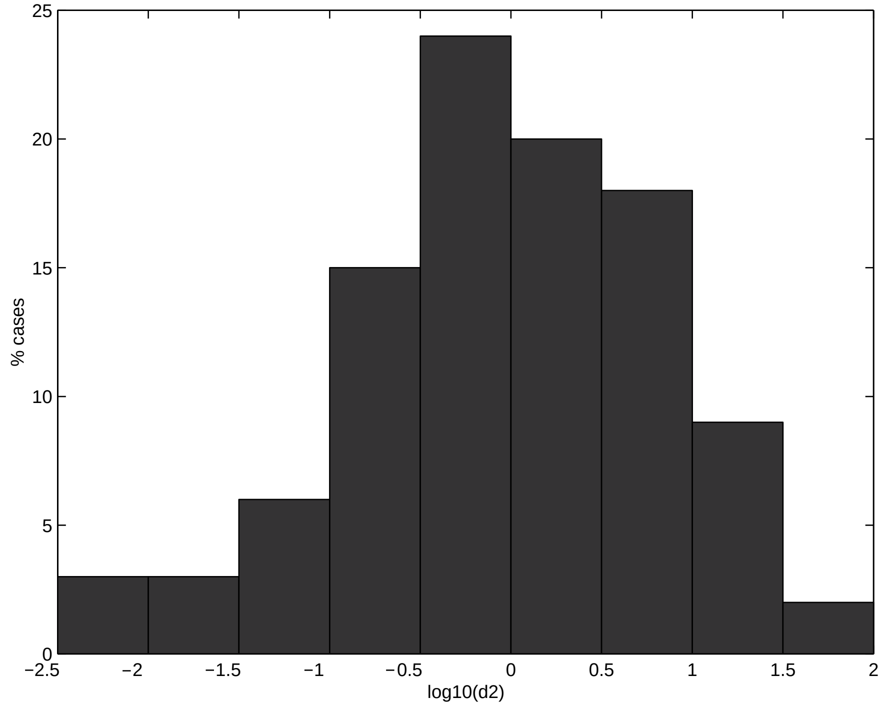

FIG. 2. Histogram of the values of the inclination-only distance d_2: logarithm in base 10 of the value versus number of cases (out of 100).

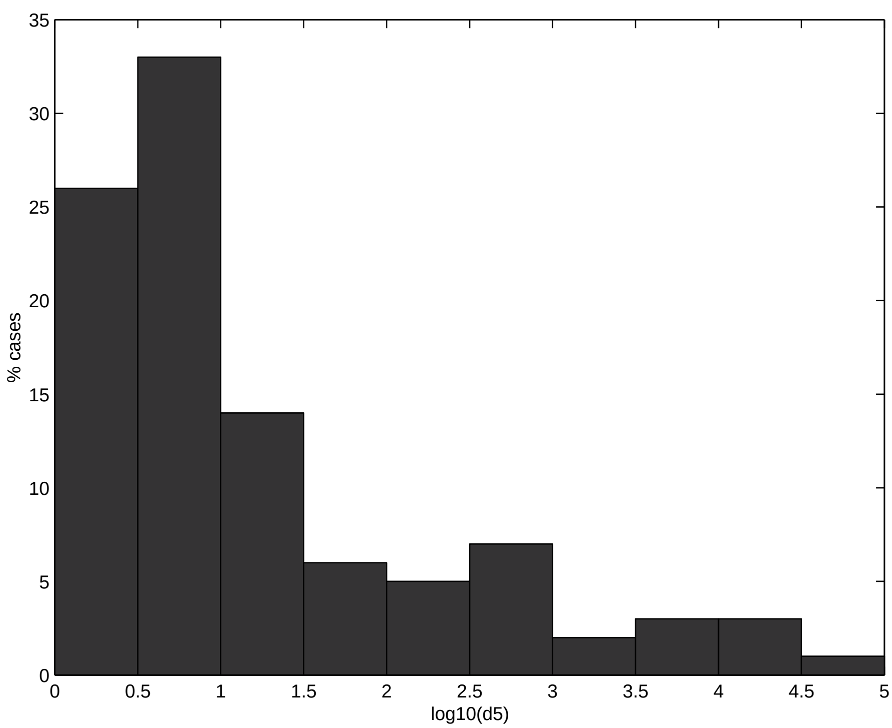

FIG. 3. Histogram of the values of the distance d_5, computed on all the elements but the mean longitude: logarithm in base 10 of the value versus number of cases (out of 100).

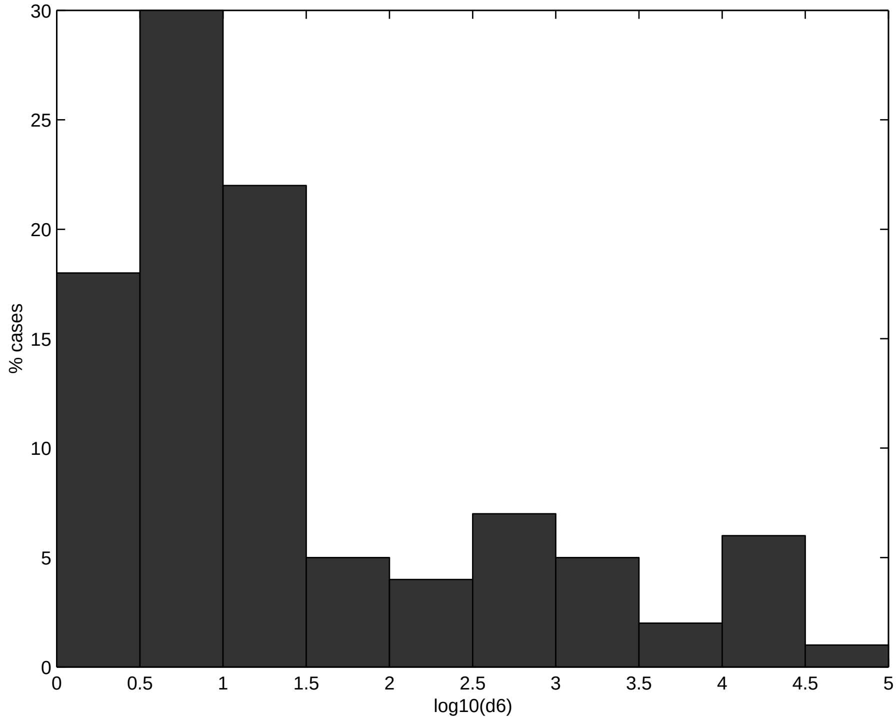

FIG. 4. Histogram of the values of the distance d_6, computed on all the elements: logarithm in base 10 of the value versus number of cases (out of 100).

identification metrics. These two formulas would give identical results in exact arithmetic, but are susceptible to giving discordant results when the normal matrices have very large conditioning numbers. The alternative values for both d_2 and d_5 are essentially identical (differences less than 0.01) in all cases, while for d_6 the differences are larger than $1/10$ of the value of d_6 in 13% of the cases. This is not unexpected: the full covariance matrix (also the normal matrix) has a conditioning number growing with the time elapsed after the observations, as it can be seen already from the two-body approximation of Paper I, Section 4.1; the reduced 5×5 matrix does not have this property. Anyway, the difference between the two values is not really important in any of the test cases.

Another important test of our identification algorithm is the following. The theory outlined in Section 2 not only provides a minimum identification penalty but also a *first guess* for the orbit resulting from the identification. The full linear algorithm computes a complete set of orbital elements (X_0 in the notations used in Section 2.1) which is the best solution in the linear approximation and should be used as a starting value for an iterative differential correction procedure, including all the observations from both arcs.

On the contrary, the restricted identification procedure provides only a first guess for the orbital elements in the selected subspace (E_0 in the notations used in Section 2.2). For the reasons already discussed, this procedure does not provide a first guess for the elements not included in the vector E. Thus, the identification based only upon the two elements p, q does not provide a useful first guess, and the procedure based upon five orbital elements provides a first guess for all the elements but the mean longitude λ.

A simple-minded procedure could be to devise some first guess λ_0 by a procedure that does not take into account the uncertainty of the two separate solutions for the two arcs, e.g., λ_0 could be just the mean of the two values resulting from the two separate arc solutions for the same epoch t_0 (note that the average of two angles is not the average of their principal values, but needs to be done with some care to avoid a mistake by $2\pi/2$).

As it could have been expected, this method of computing a first guess is not very successful: the differential correction iterative procedure converges to the identification orbit for only 80% of the cases. But it can be valuable in cases where the d_6 is not useful due to very large uncertainty in λ. In fact, if the marginal uncertainty in λ is more than one revolution the d_6 computation can actually return more than one value, because the difference in λ can contain integer multiples of 2π. In these cases we use the result with the lowest d_6, but the reliability of the test in these cases is dubious.

Then the real test of the quality of the linear identification algorithm is to try to achieve convergence of the iterative differential corrections procedure, using as a starting point the X_0 set of orbital elements suggested by the algorithm as best solution in the linear approximation. We have performed this test and found convergence to the identification orbit in 99% of the cases.

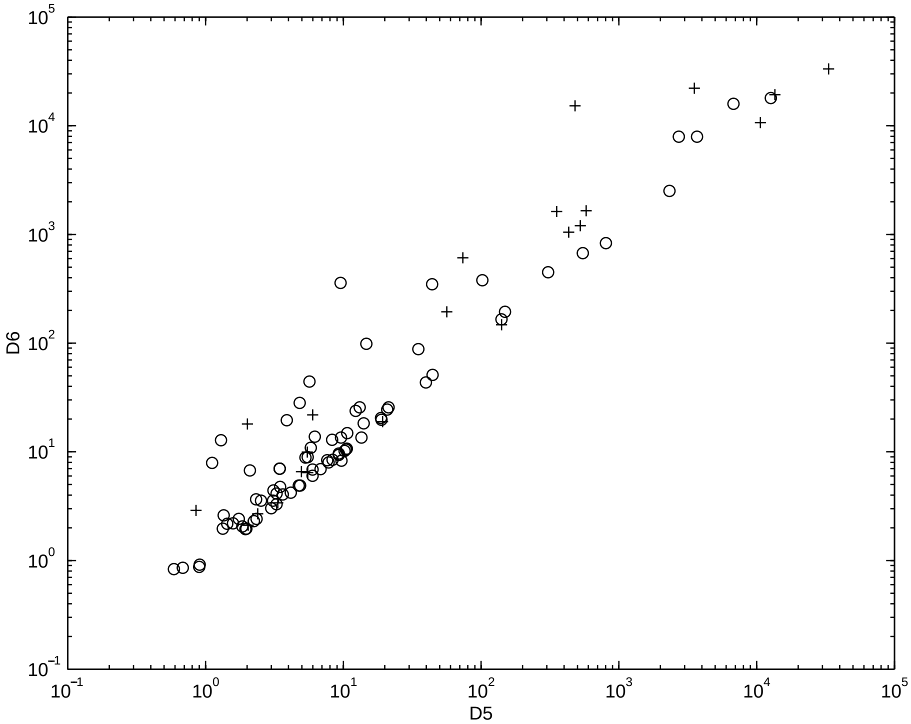

FIG. 5. Values of the identification metrics d_5 versus d_6 in the 100 test cases. The circles indicate the 80 cases with convergence of differential corrections starting from the 5×5 first guess; the crosses indicate the cases where convergence has been obtained, but only from a better first guess.

The results of these tests on the convergence of differential corrections are summarized in Fig. 5. The crosses indicate the cases in which the convergence failed, when starting from the E_0 guess; in all these cases, however, the full algorithm starting from X_0 succeeded, with only one exception, which is indicated by the cross at the top right of the figure, with values of both d_5 and d_6 above 30,000. Even in this isolated case, the differential corrections algorithm converged when the first guess was the set of interpolated elements obtained with the algorithm proposed in Sansaturio *et al.* (1996).

5. NEW IDENTIFICATIONS

After a hiatus of several years, the global dataset of astrometric observations of asteroids was made available to the scientific community in January 1999. This has permitted us to test the theory described in this paper more realistically than the experiments of the previous section. In this section we outline our procedure to find new orbit identifications starting from this dataset and the results we have obtained.

We have used the dataset, available from the Minor Planet Center (MPC), containing all the published asteroid observations. This dataset is currently updated near each full moon, and we have used the March 2, 1999, and the April 2, 1999, updates in our testing. In the following discussion all numbers refer to the April catalog unless stated otherwise.

To give an idea of the size of the archive of observations, consider that the dataset for only the unnumbered asteroids contains 1,157,884 observations for 121,090 designations. This does not imply that there are really more than 100,000 distinct asteroids that have been observed, but only that there have been that many separate discoveries. In fact, there are 15,461 (secondary) designations belonging to objects that have been identified with 11,231 other (primary) designations. Note that these identifications do not necessarily lead to *multiopposition* orbits, because sometimes two sets of observations belonging to the same opposition/apparition are identified (the MPC uses the specific term *double designations* for these cases). These numbers of identifications refer to the April situation; thus they already include identifications that we had ourselves proposed in March and which had already been processed by the MPC. There are 94,398 designations that have never been identified with another.

The first step is to compute a catalog of orbits, complete with normal and covariance matrices, but it is neither possible nor useful to compute orbits for each one of the $94{,}398 + 11{,}231$ “asteroids” in the files. There are 8,367 identifiers corresponding to a single observation, and these are essentially useless. There are also 8,870 identifiers corresponding to two observations, and some of these can be used for attributions, as we will discuss in a later paper in this series; however, they cannot be used to compute a full orbit with six independently solved for orbital elements. There are another 31,946 identifiers corresponding to at least three observations, which, however, span less than 4 days; for many of these an orbit could be computed, but it would be very poorly constrained. All the difficulties described in Section 3 would be very severe for such very short arc orbits, and methods more suitable to strongly nonlinear identifications are necessary. An additional difficulty arises from observations that have been reported only as rough positions; an arc including less than three "good" observations might result in a nominal orbit, which is however of little significance. Finally, no quality control can be performed on arcs containing only the minimum number of three observations, and residual normalization is meaningless.

For these reasons we have selected only the 35,857 objects for which there are at least four good observations and with arcs of at least 4 days. Of these, there were 737 objects for which we could not compute an unconstrained orbit with our automated orbit determination software. So we have computed 11,875 orbits with observed arcs longer than 180 days, 11,399 orbits with arcs between 20 and 180 days, and 11,846 orbits with arcs between 4 and 20 days. We have thus assembled a catalog with 35,121 orbits (including *(719) Albert*, the only lost numbered asteroid). Each of these orbits has been computed as the solution of a least-squares fit with convergent differential corrections. The automated outlier rejection used in this process will be described in another paper of this series, but in practice the control parameters were such that the outlier removal was inactive for short arcs and quite effective for multiopposition orbits. The residual normalization was applied by using the maximum between 1 arc sec and the actual residuals RMS. Thus normal and covariance matrices were available for each orbit.

To minimize the effect of nonlinearity in the propagation of the confidence regions, we have implemented a method whereby we can access different catalogs at several different epochs, in order to use for each couple being tested the epoch closest to the midpoint of the two central epochs. This results in a measurable, but not dramatic (about 10%), increase of the number of real identifications found.

To apply the algorithm described in Sections 2 and 4, we have to perform $35{,}121 \times 35{,}120/2$ computations of the orbit plane distance d_2; 3,895,552 couples passed the test $d_2 < 30$, for these the distance d_5 (based upon all elements but λ) was computed, and 300,977 passed the test $d_5 < 5{,}000$. For the latter, the full linear identification distance d_6 was computed, and $d_6 < 100{,}000$ was satisfied by 105,853 cases. At this point the output file was sorted by the value of d_6 (for example, in the April run there were 2,337 cases with $d_6 < 1{,}000$ that appear promising given the results of the tests of Section 4) and *identification check* runs were started. Each identification check consisted of an iterative differential correction procedure, attempting to fit the observations of both orbits to a single orbit, starting from the first guess X_0 computed with the full linear identification algorithm. During this procedure the automated outlier rejection was turned off to avoid the case—which indeed can occur—that most of the observations from one of the two arcs are rejected; the outliers already removed in the fit of each of the two separate arcs were left out.

The number of cases passing each test in the preceding paragraph are from one particular run performed during the April update and are given only as an example. In fact we have run the programs numerous times, experimenting with slightly different values of all the controls. The procedure is analogous to the sifting of tons of sand and gravel to find a few gold nuggets. However, the difficulty is not in shoveling tons of gravel: today's computers are so powerful that this amount of data processing (e.g., $\simeq$600,000,000 computations of d_2) requires negligible resources (we only have Pentium-based PCs). The main challenge is in achieving full automation of the procedure and in guaranteeing a very tight quality control.

To stress the importance of high quality work, and before evaluating the practical results, that is, the new identifications that we have actually found in this way, we need to point out one main conceptual difference between our search for orbit identifications and the gold mining analogy. Our work is more like the sifting done by todays' tourists, who are allowed to rescan the refuse dumps of the gold rush ghost towns. In fact the data we receive from the MPC have already been scanned for identification by the MPC itself, and to the extent that some information on these data was available before, also by other identification diggers. That we could have found the same identifications found by others has been shown in Section 4, but this is not the point. The big, shiny gold lumps have been found long ago; our methods have to be so much more sophisticated that the identifications that have already escaped all the other methods of detection can be found.

Table I contains a summary of the orbit identifications we proposed to the MPC. These are the cases for which we could find a common orbit, to which the observations of both arcs could be fitted with reasonably small (less than $\simeq$1.4 arc sec) RMS without additional outlier removals. Nevertheless, some of these fits did show systematic errors in the residuals and were therefore rated either *marginal* or *poor* identifications after visual inspection of the residuals with a simple graphics program; some of the marginal and poor cases have not been accepted as identifications by the MPC, but all the cases we

TABLE I
Summary of Orbit Identifications

	Good	Marginal	Poor	Total
March 1999 dataset				
Submitted to MPC	104	23	6	133
Published by MPC	104	19	2	125
Credited to us	76	15	1	92
April 1999 dataset				
Submitted to MPC	14	13	6	33
Published by MPC	14	12	1	27
Credited to us	11	8	1	20

TABLE II
Published Identifications Obtained with Our Method

Credited to:	Us	Others
With 1999 designations	17	30
With 1998 designations	51	8
Earlier designations	44	2
Total	112	40

rated *good* have been accepted. Some of the orbit identifications we have proposed have not been published by the MPC under our names, even though they were accepted, because somebody else had proposed them already. Note that this happened in the time span between when the observational data update was made available by the MPC and the date of our submission, that is less than 2 weeks. This gives an idea of the tight, and indeed stimulating, competition to find asteroid identifications.

The decline in submissions in Table I between the March and April updates is due to the fact that the method is new, and there is an initial cleanup with a computational cost that is quadratic in the number of objects tested. We are still working to find optimum filter parameters, so the cleanup continues at a much slower pace; however, at some point the process will switch to a maintenance mode where only objects that have had new observations (or identifications) in the previous month need to be tested. In this mode the computational expense is only linear in the number of objects to be tested.

To get a feeling about where the present method is most successful, consider Table II. Here we distinguish between the 40 identifications that we submitted to the MPC that were credited to *others* and the 112 identifications that were not found by others. Most of the 112 credited discoveries do not include very recent (1999) designations. Conversely, almost all of those that were independently discovered by others are associated with the more recent designations. It is true that the table only reflects those identifications which were obtained from our method, and the 40 that have been discovered by others should not be considered a representative sample of the work done by others in the field. This is especially true in light of the fact that the vast majority of identifications published by the MPC are discovered on the basis of attribution of observations rather than identification of orbits. But, on the other hand, the data from the table indicate that our method is capable of finding some "difficult" identifications that have been hiding in the catalogs for a long time.

Table III lists a random sampling of the 152 published identifications that were obtained with the present method, though not all of these have been credited to us. A full list of all the orbit identifications that we have proposed can be found online at `http://copernico.dm.unipi.it/identifications/`.

One important parameter to evaluate the "difficulty" of an identification is the distance of the nominal orbit solutions $X_i = (a_i, h_i, k_i, p_i, q_i, \lambda_i)$, $i = 1, 2$ being identified. We have sorted Table III by a simple distance given by

$$d = \sqrt{\left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 + (h_1 - h_2)^2 + (k_1 - k_2)^2 + (p_1 - p_2)^2 + (q_1 - q_2)^2},$$

and it is clear that a significant fraction have large d (40 out of 152 accepted identifications have $d > 0.1$).

There are some identifications in Table III with high values of d_6; the reason for this is that some of the orbit identifications have been proposed because they had low d_6, while some had been selected for confirmation by sorting on d_5. The value of d_6 is most subject to numerical instabilities and to the nonlinearity effects; thus, a low d_5 couple might be worth checking even if the value of d_6 is large.

Many of our proposed identifications involved asteroids recently discovered. This is just the "new lode" effect; that is, these orbits had been subjected to a less extensive search for identifications. However, some of these are still quite interesting because of the very long time span between the two observed arcs; for example, we have been credited with discovering three identifications that link asteroids originally discovered during the 1960 Palomar–Leiden survey to objects discovered in 1999. Since many new asteroids are discovered every month, we can expect to continue to find such new cases after each monthly update.

Some of the cases, on the contrary, concerned only orbits of asteroids discovered a long time ago. For example, 1510T-2 = 1283T-1 and 1232T-3 = 1056T-1 are identifications of asteroids found in the Trojan surveys T-1 (of 1971), T-2 (of 1973), and T-3 (of 1977): they were not suspected to be the same by the Trojan surveyors, and then they escaped the attention of the MPC and of all the other identification diggers for decades. In 15 of the 152 published identifications both components were discovered before 1995, and all but one of these cases were credited to us. We should not expect to find many more of these "nuggets in the dumps" in the future, unless we further improve our methods, which is, in fact, a work in progress.

6. CONCLUSIONS AND FUTURE WORK

From the test performed in Section 4 we can only conclude that, of the already known identifications, 99% could have been found by using the algorithm of linear identification (as described in Section 2.2) to provide a first guess, followed by standard iterative differential corrections. Even the remaining 1% could have been found at the cost of performing two attempts of differential corrections, using an alternative (low computational cost) algorithm to compute the first guess. Thus, there is good evidence that we have found a workable algorithm to propose orbit identifications.

The question is, how effective such an algorithm is in finding new, not previously proposed identifications. The results we have obtained in the March and April 1999 runs, essentially 152

TABLE III
Sample Orbit Identifications

Designation 1	Designation 2	d	d_2	d_5	d_6	RMS of residuals
1997GK16	1989UK7	0.324	0.35	2.27	512.58	0.51
1997CE8	1981PP	0.204	0.07	1800.95	3850.80	0.55
1996YP2	1992RS5	0.194	1.91	95.32	5713.94	0.50
1232T-3	1056T-1	0.159	0.14	161.19	406.52	0.75
1999CT59	1988CY	0.147	1.89	350.39	402.40	0.57
1999CY34	1990SQ7	0.138	6.07	39.87	2005.11	0.77
1999CG86	1993ON8	0.121	4.04	383.90	2004.78	1.07
4868P-L	1997TT11	0.114	0.25	0.74	13848.41	0.28
1998XD29	1997NP	0.111	0.56	6.26	10.73	0.56
1998ST66	1993RG12	0.105	2.90	8.12	8.41	1.18
1997WL37	1993RF4	0.099	17.60	89.70	372.88	1.13
1999FA19	1995DV	0.092	2.89	295.36	305.50	0.64
1131T-2	1998JJ2	0.085	7.39	27.10	7084.67	1.15
1998FS15	1990KF1	0.080	2.32	273.87	278.35	1.41
1998FV87	1978VM9	0.074	0.65	8.30	10.62	1.64
1996VA30	1981UU	0.065	7.61	194.83	891.30	1.27
1998HH52	1983VF7	0.059	1.09	10.92	45771.18	0.81
1998SE7	1994NA3	0.053	0.42	4.64	23.50	0.75
1999CN47	1989GB2	0.047	0.13	13.40	15.65	0.51
1979QZ3	1993QD4	0.042	15.90	1790.95	3172.75	1.25
1999CA73	1993US2	0.040	0.98	34.15	51.42	0.96
1999BP9	1991LN	0.038	0.62	12.07	12.42	0.74
4079T-3	1998VK13	0.034	2.70	4.04	14.64	0.98
1999BG11	1980FC4	0.033	0.99	4.40	4.50	0.59
1998UF40	1991RR26	0.031	0.15	1.90	5.65	0.61
1510T-2	1283T-1	0.029	0.05	7.74	7.87	1.10
1998XW77	1996EP1	0.028	0.05	3.93	4.58	0.70
1999EA5	1997TM26	0.024	1.95	37.09	45.02	1.41
1997TW21	1996HW22	0.023	1.02	83.46	90.35	1.50
9566P-L	1999CN25	0.019	0.25	0.75	40.55	0.57
2234P-L	1998SG8	0.016	0.53	1.01	7.02	0.64
1997MC	1994WP13	0.013	1.11	8.75	10.23	0.92
6705P-L	1998FH6	0.011	0.71	2.15	3.51	0.68
1997WP	1978UE5	0.009	3.48	9.05	9.80	0.77
1999CF66	1996HJ24	0.007	0.77	6.87	6.95	0.61
1993PD7	1970PU	0.005	9.88	14.05	14.46	1.54
2039P-L	1998RZ59	0.002	1.52	3.07	2.10	0.71
4218P-L	1998FF103	0.001	0.16	0.32	0.36	0.44

"true" orbit identifications, are impressive and at the same time inadequate. More than 100 of these were not based on recent data. They are impressive in that we have recovered 152 lost asteroids, and this by pure computation, without using a telescope; moreover, we have removed from the catalogs 304 poor orbits, for which telescope recovery would have been time-consuming, in some cases almost impossible. But they are inadequate to the size of the problem in that the number of orbits has only decreased by less than 1%. Thus we can conclude that yes, the algorithm of this paper is effective in detecting orbit identifications, which not only were unknown, but in many cases did escape to all the other methods in use for identification. We have also to conclude that no, the method of this paper is not the final solution of the orbit identification problem, because many more are certainly yet to be found.

The conclusion of this paper should therefore be positive: this method should be used because it is very effective, especially in that it uses efficiently both computing and telescope resources. It needs to be stressed that this efficiency would be even more significant if this method was used as a matter of routine on all new discoveries for which an orbit can be computed (by least-squares fit of all the elements). In this case the computational complexity would be linear in the number of known orbits, not quadratic as it happens in an initial catalog cleanup with a new method. The immediate availability of the identified orbit, as a result of a fully automated procedure, without waiting for the identification diggers' hard work, allows us to immediately exploit it for attributions and close approach analysis; dedicated astrometric follow up may become unnecessary, and the reduced uncertainty makes telescope recovery much easier in case it is needed.

The conclusions about the global problem of asteroid identification are not so simple, and although they are beyond the scope of this paper, we need to anticipate some of them, at least as much as it is necessary to understand the direction of our future work.

One possible way to attack the problem, that is, to find many more identifications, is to take into better account the nonlinearity of the identification problem. There is a well-known way to approximate a linear function better than a single linear approximation: to use the differential, namely, the local linear approximation, at many different points. This can be achieved by combining the multiple solutions algorithm of Paper I, Section 5, with the linear algorithm of this paper, Section 2.

Another direction is to look for identifications of the attribution type, that is, to perform matching of the observations on the celestial sphere rather than of the elements in the phase space. It is indeed the case that today many more identifications are found by attribution, rather than by orbit identification, and this by many people, including ourselves. Attribution methods are thus promising, but they also have their problems, especially in the difficulty of confirming the identification without additional observations.

In the near future we plan to work following these two approaches and also by combining both of them.

The software to compute all the identification metrics and the first guess for identification, as discussed in this paper, has been included in the free software system ORBFIT. The most recent version of ORBFIT (currently 20.0) as well as online documentation can be obtained on the WWW at `http://newton.dm.unipi.it/˜asteroid/orbit/`. It is also available by anonymous ftp from the server `newton.dm.unipi.it`, in the directory `pub/orbfit`.

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REFERENCES

Cappellari, J. O., C. E. Velez, and A. J. Fuchs 1976. *Mathematical Theory of the Goddard Trajectory Determination System*, 3 Vols. Goddard Space Flight Center X-582-76-77.

Gauss, C. F. 1809. *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections*. Reprinted by Dover, New York, 1963.

Milani, A. 1999. The asteroid identification problem. I. Recovery of lost asteroids. *Icarus* **137**, 269–292.

Milani, A., and G. B. Valsecchi 1999. The asteroid identification problem. II. Target plane confidence boundaries. *Icarus* **140**, 408–423.

Sansaturio, M. E., A. Milani, and L. Cattaneo 1996. Nonlinear optimisation and the asteroid identification problem. In *Dynamics, Ephemerides and Astrometry of the Solar System* (S. Ferraz Mello, B. Morando, and J.-E. Arlot, Eds.), pp. 193–198. Kluwer Academic, Dordrecht/Norwell, MA.