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The identification problem in asteroid surveys

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Abstract. The statistics determining the strategies for identifying the great number of new asteroids in surveys is discussed. If computed positions are tentatively identified with the nearest trails, then the probability function for the number of errors is geometric, with the parameter depending on the average number of trails inside the error ellipse of the prediction. If positions are transformed to the slowly varying parameters T and I, then interpolation methods may give higher accuracies in the linking of two lunations than computations based on perihelion orbits.The most probable position predicted from two positions is given by the "maximum likelihood orbit".

Key words: methods: data analysis – surveys – astrometry – celestial mechanics – minor planets

1. Introduction

A practical problem in asteroid surveys is to follow the great number of unknown main belt objects. The main problem is to link the two periods of observation interrupted by a full Moon. Although the ultimate goal is to derive reliable elements, the objects must for a few days be followed by linear extrapolation. An interesting theoretical question is to determine *when* it will be most economical to shift from interpolation methods to orbit computations. Our main purpose here is to demonstrate, that laborious computations of ill–determined initial elements will not be needed, and orbit computations may be postponed until they make sense.

The density of new objects per square degree and the accuracy of predicted positions is combined in Sect. 2 into a critical parameter which determines the probability of confusing different objects. A rational follow up procedure combines a well distributed set of observations with this statistics.

The individual motions and their scatter are derived in Sect. 3. This, and the mean distance between the objects, determines the time of mixing of the crowd of trails. That different objects are likely to have different daily motions is also the basis of the general use of position *pairs* for identification. The formulae of this section give as a secondary result that in case a circular orbit is not possible, the most probable orbit is a perihelion orbit ($r < a$).

Sec. 5 considers two–dimensional motion in (α, δ) and in the time of opposition (T) and minimum inclination (I) introduced by Kristensen (1990); for two days the motion can be considered linear in (α, δ) and for a week in (T, I).

The last Sect. 7 demonstrates that interpolation in (T, I) can bridge a 40 day gap with the small errors $\pm 4'$ and $\pm 1'$. This is achieved by a determination of fictitious intermediate places independently from *all* positions in the first and second lunation. By using all information simultaneously, the new method differs from the familiar prediction of *one* of the observations in the second lunation by an orbit based on all observations in the first.

2. Andrae's error ellipse and statistics

Consider a field with a density of μ randomly scattered objects per square degree. The problem is to select a specific object on the basis of an uncertain prediction.

In the Palomar–Leiden Survey was found 10.9 objects per square degree (Fig. 1 in van Houten et al. 1970) brighter than $B \lesssim 20.5$. With the mass population index 11/6 of Dohnanyi (1969) the mean number of asteroids per square degree brighter than B becomes

$$\mu = 10^{(B-18.4)/2} \tag{1}$$

This expression illustrates immediately that the magnitude is not useful to discriminate between faint objects. If objects 2 magnitudes brighter than the one in question are deleted in a search, the number of candidates is only reduced by 10%. Magnitudes may be of help only for the few brightest objects.

Suppose an object must be inside a given area A square degrees. The probability that there are by pure chance n foreign objects in this area is given by the Poisson distribution

$$\frac{(\mu A)^n}{n!} e^{-\mu A} \tag{2}$$

Rather than a definite area, predictions will give a probability density, which often may be approximated by a two–dimensional Gauss distribution. Let X and Y denote the coordinates (plate coordinates, $\cos\delta\Delta\alpha, \Delta\delta, \cdots$ etc.) and assume for simplicity that the axes coincide with the main axes of the statistical distribution. The components of the prediction are then independently Gauss distributed with means X_0 and Y_0 and mean errors σ_X and σ_Y. The prediction is simply written

$$X = X_0 \pm \sigma_X, \qquad Y = Y_0 \pm \sigma_Y \tag{3}$$

Let χ measure the distance between a trail and the predicted position

$$\chi^2 = \left(\frac{X - X_0}{\sigma_X}\right)^2 + \left(\frac{Y - Y_0}{\sigma_Y}\right)^2 \tag{4}$$

The observed trails are ordered according to increasing distances from the predicted position. The identification procedure consists in trying to identify the predicted position successively with the ordered trails. If the nearest trail is proved inconsistent with the observations the next nearest is tried, and so forth. The process is described by the statistical distribution of the number n of foreign objects inserted accidentally between the predicted and the true positions.

For $\chi^2 = 2$ formula (4) gives the error ellipse, which important concept was introduced in connection with the Danish Geodetic Survey by minister C.C.G. Andrae (1867). He chose an ellipse inside which the probability is 63%, which is conveniently comparable to the 68% of the familiar 1σ limit. In the present problem the mean number (p) of random objects in Andrae's error ellipse is a very important parameter:

$$p = 2\pi\mu\sigma_X\sigma_Y \tag{5}$$

The probability dP_n, that the predicted object is found in the interval χ to $\chi + d\chi$, and that by chance there are n foreign objects between the predicted and true position, is the product of the two–dimensional Gauss distribution and (2) (with $\mu A = p\chi^2/2$)

$$dP_n = \frac{(p\chi^2/2)^n}{n!} e^{-\frac{p\chi^2}{2}} e^{-\frac{\chi^2}{2}} \chi d\chi \tag{6}$$

Integration from 0 to χ^2 gives

$$P_n(\chi^2) = \frac{p^n}{(1+p)^{n+1}} P\left((1+p)\chi^2 | 2n+2\right) \underset{\chi\to\infty}{\longrightarrow} \frac{p^n}{(1+p)^{n+1}} \tag{7}$$

expressed in terms of the familiar χ^2–distribution (Abramowitz et al. 1964) defined by

$$P(\chi^2 \mid \nu) \equiv \frac{1}{2^{\nu/2}\Gamma(\nu/2)} \int_0^{\chi^2} t^{\nu/2-1} e^{-t/2} dt \tag{8}$$

In the limit $\chi \to \infty$ the probability of n foreign objects converges to the simple geometric distribution with mean p and variance $p(p+1)$.

The probability that the trail nearest to the predicted position is the correct identification is given by (7) with $n = 0$.

$$P_0 = 1/(1+p) \tag{9}$$

Preferably this probability should be large, but a rational follow-up strategy is designed to minimize the total effort needed to achieve orbits with the prescribed quality. Plates, secured initially in rapid succession in order to maintain the identities of new discoveries, will not improve the final orbits, and the rational computing procedure is not necessarily the one with the highest accuracy ($p \approx 0$), but the one which minimizes the total computing time.

Let a computation of orbital elements require the computing time τ. The elements may give accurate predictions ($p \approx 0$) which lead to new identifications on the basis of which the orbit is improved. The total computing time per object is then 2τ. A simple extrapolation procedure, which requires negligible computing time, may, however, be preferable, if the computing time of the final orbit *plus* p mistakes, $(1+p)\tau$, is less than 2τ.

The strategy above can easily be improved and the limit $p < 1$ increased. Instead of processing all objects in the neighbourhood of a prediction until the object is found, only a single trial is made; the failures are searched for in following runs where the successful identifications are deleted.

Let N_m denote the number of objects with $m = 0, 1, 2, \cdots$ trails between the actual and predicted positions. The total number of objects is $N = \sum_0^\infty N_m$, of which only N_0 are immediately identified with the nearest position. A search run requiring the computing time $N\tau$ identify a fraction $N_0/N = 1-\alpha$ of the objects. The identified objects are removed more or less at random, and we shall derive the new distribution N'_m.

For $N_{m+1+\nu}$ objects one trail is deleted because it proved unsuccessful in the previous run, and among the remaining $m+\nu$ objects the probability is $\binom{m+\nu}{\nu}(1-\alpha)^\nu\alpha^m$ that ν objects are identified and removed. The new distribution is then given by

$$N'_m = \sum_{\nu=0}^{\infty} \binom{m+\nu}{\nu} \alpha^m (1-\alpha)^\nu N_{m+\nu+1} \tag{10}$$

The successive distributions are conveniently derived from generating functions. N'_m is the coefficient to t^m in the expansion of

$$\sum_{m=0}^{\infty} N'_m t^m = \sum_{m=1}^{\infty} N_m(\alpha t + 1 - \alpha)^{m-1} \tag{11}$$

Independent of the value of α, this gives for $t = 1$ the evident formula

$$\sum_{m=0}^{\infty} N'_m = \sum_{m=1}^{\infty} N_m \tag{12}$$

In general the process starts with the geometrical distribution (7), and the following distributions are determined recursively by (11). For the geometric distribution the generating functions can be evaluated, and N'_m also becomes a geometric distribution, with parameters $p \to p' = \alpha p$.

Table 1. Table 1 shows a fictitious numerical example, starting with $N = 1000$ objects and a geometric distribution with the critical parameter $p = 1.54$. In the first search run all the 1000 predicted positions are tentatively identified with the nearest trail. A fraction $\alpha = p/(1+p) = 0.606$ of these identifications are failures, so 606 objects remain for the following run. The second distribution is again geometric but the smaller density of trails gives the smaller critical parameter $p' = \alpha p = 0.934$ and $\alpha' = 0.483$. The second run leaves 293 unidentified objects and so on. The table illustrates the fast convergence due to the decreasing density. The average number of orbits computed per object is 2.00 which shows that p should no exceed 1.54.

Run	p	α	N
1	1.54	0.606	1000
2	0.934	0.483	606
3	0.451	0.311	293
4	0.140	0.123	91
5	0.017	0.017	11
		Sum:	2000

Table 1 illustrates this by a hypothetical example with $N = 1000$ and $p = 1.54$. The fraction α of failures decreases rapidly in the five search runs due to removal of established identities. The average number of orbits computed per object is 2.00. Although of little practical interest, the process converges formally for large values of p. Approximate expressions for the number of orbits per object are:

$$p/2 + 1.430 - 0.423/p \quad (p > 1.5), \quad 1 + p/(1+p)(p \lesssim 0.2) \tag{13}$$

3. Individual motions and their dispersion

In heliocentric spherical coordinates with longitude l, latitude b and radius vector r, components of velocities $\dot{r}$, parallel ($v_{\neq}$) and orthogonal ($v_{\perp}$) to the ecliptic plane are

$$\dot{r} = \frac{r\dot{v}e\sin v}{1 + e\cos v} \tag{14}$$

$$v_{\neq} = r\dot{v}\cos i' \tag{15}$$

$$v_{\perp} = r\dot{v}\sin i' \tag{16}$$

where v is the true anomaly and i' is the inclination of the orbit relative to the parallel circle at latitude b

$$\cos i' = \cos i/\cos b, \quad \sin i' = \sin i \cos(l - \Omega). \tag{17}$$

The transverse velocity is derived from angular momentum

$$r^2\dot{v} = k\sqrt{p}, \; p = a(1-e^2) = r(1 + e\cos v), \; k = 0.9856°/d. \tag{18}$$

The distribution of the individual velocities in a given region is to first order determined by the statistical distribution of eccentricity (e), perihelion ($\tilde{\omega}$), inclination (i), and node (Ω). The statistical tables of Gondolatsch (1965) indicate that these elements are approximately Gauss distributed

$$\left.\begin{matrix} e\cos\tilde{\omega} \\ e\sin\tilde{\omega} \end{matrix}\right\} = 0 \pm 0.120 \quad \text{(m.e)} \tag{19}$$

and

$$\left.\begin{matrix} \sin i\cos\Omega \\ \sin i\sin\Omega \end{matrix}\right\} = 0 \pm 0.132 \quad \text{(m.e.)} \tag{20}$$

These distributions are assumed independent of the semi-major axis **a**. This can only be true for "proper elements", but for the present purpose we ignore this refinement, and that the perihelion of Jupiter is a preferred direction for $\tilde{\omega}$.

If λ is the longitude in orbit, or any other parameter, then the two linear combinations

$$\sin i\cos(\lambda - \Omega) = \sin i', \; \sin i\sin(\lambda - \Omega) = \sin b \tag{21}$$

will by (20) be independent random variables. The same is true for $e{}^{\cos}_{\sin}v$, where $v = \lambda - \tilde{\omega}$ is true anomaly. To first order in e and i is obtained

$$r\dot{v} = \frac{k}{\sqrt{r}}\sqrt{1 + e\cos v} \approx \frac{k}{\sqrt{r}} + \frac{k}{2\sqrt{r}}e\cos v \tag{22}$$

which gives for the velocity components (14)–(16)

$$\dot{r} = 0 \pm 0.120\frac{k}{\sqrt{r}}, \; v_{\neq} = \frac{k}{\sqrt{r}} \pm 0.060\frac{k}{\sqrt{r}}, \; v_{\perp} = 0 \pm 0.132\frac{k}{\sqrt{r}} \tag{23}$$

An immediate, but useful, application is to compute the root–mean–square relative velocity between two, possibly colliding, planets at distance r

$$k\sqrt{\frac{2}{r}(\frac{5}{4}0.120^2 + 0.132^2)} = 4.79 \text{ km/s for } r = 2.74\text{AU} \tag{24}$$

Our primary interest is, however, to obtain the dispersion in the motion in geocentric longitude (λ) and latitude (β)

$$\dot{\lambda}\cos\beta = \frac{k\left(\frac{\sqrt{a}}{r}\cos i' - 1\right)}{r-1} + \frac{\dot{r}\sin E}{r(r-1)} \tag{25}$$

$$\dot{\beta} = \frac{k\sqrt{a}\sin i'}{r(r-1)} - \frac{\dot{r}\sin\beta\cos E}{r(r-1)} \tag{26}$$

where E is the elongation $\lambda_{\odot} - 180° - \lambda$ and $\dot{r}\sin E$ is the projection of $\dot{r}$ on the sky plane. The second order term $\dot{r}\sin\beta$ may be ignored.

The motion in longitude (25) is the sum of 3 components. With the probability distribution of the distances from Table 1 in Kristensen (1990) the mean value and scatter of the individual terms become

$$k(r^{1/2} - 1)/(r-1) = (-0.225 \pm .0225)°/d \tag{27}$$

$$kr^{-1/2}e\cos v/2(r-1) = (0 \pm 0.0220)°/d \tag{28}$$

and

$$-kr^{-3/2}e\sin v\sin E/(r-1) = (0 \pm .0174\sin E)°/d \tag{29}$$

The total motion in longitude becomes

$$\dot{\lambda}\cos\beta = (-0.225 \pm 0.0315 \pm 0.0174\sin E)°/d \tag{30}$$

The mean daily motion in the elongation E is $1.211°/d$ (= .9856+ .225); within a month from opposition the $\sin E$-term contributes little to the scatter.

For the motion in latitude (β) is similarly obtained

$$\dot{\beta} = (0 \pm 0.0485)°/d \tag{31}$$

The scatters in the daily motions, (30) and (31), have two applications:

The first is to obtain the maximum time–difference $t = t_1 - t_0$ for which two positions may be identified immediately by inspection. From positions at time t_0 positions are "predicted" at time t_1 by the mean motion. The mean errors in longitude and latitude of these "predictions" are respectively $0.^\circ0315t$ and $0.^\circ0485t$. With an assumed density $\mu = 20$ (corresponding to $B \lesssim 21.0$) and a probability $1/(1+p)$ better than 99% for the correctness of the identification with the nearest trail, is required

$$p = 2\pi\mu\sigma_\lambda\sigma_\beta = 2\pi \cdot 20 \cdot 0.0315 \cdot 0.0485 \cdot t^2 \lesssim 0.01, \quad \text{or } |t| < 0.^{d}23. \tag{32}$$

Already after one day many objects will be confused, and in order to maintain their identity continuously, pairs of positions or ends of trails must be available from the same night.

Larger time differences will mix the objects together and require an analysis of at least one third epoch plate by which objects showing a linear motion can be identified.

The second example is the use of pairs of positions for the verification of identities. This is a classical and often used method of great value; among others are the examples of the identification of lost planets with positions on two consecutive nights (Kristensen 1981).

Given two observed positions of the same object at times t_0 and t_1 which are suspected to belong to a somewhat uncertain ephemeris. An ephemeris correction is applied which merge the ephemeris with the observed position at time t_0. The improved ephemeris at time t_1 is then compared with the observation. If the observations and the ephemeris refer to two different sets of elements, which here may be considered selected at random, the *difference* between their positions at time t_1 is

$$\cos\beta(\lambda' - \lambda) = 0 \pm \sigma_\lambda,\ \sigma_\lambda = \sqrt{2}\, 0.0315t,\ t = t_1 - t_0. \qquad (33)$$
$$(\beta' - \beta) = 0 \pm \sigma_\beta,\ \sigma_\beta = \sqrt{2}\, 0.0485t$$

The probability that by chance

$$\left(\frac{\cos\beta(\lambda' - \lambda)}{\sigma_\lambda}\right)^2 + \left(\frac{\beta' - \beta}{\sigma_\beta}\right)^2 \leq \chi^2 \qquad (34)$$

is given by (7)

$$P_0(\chi^2) = \left(1 - e^{-\chi^2(1+p)/2}\right)/(1+p) \approx \frac{\chi^2}{2} \qquad (35)$$

The probability is thus less than 1% for having by pure chance an agreement better than within an ellipse with axes

$$\sqrt{0.02} \cdot \sqrt{2} \cdot 0\overset{''}{.}0315t = 23''t \qquad \sqrt{0.02} \cdot \sqrt{2} \cdot 0\overset{''}{.}0485t = 35''t \qquad (36)$$

The time interval t for a valid identification must be chosen large enough to make these axes significantly larger than the measuring and, formerly, the ephemeris rounding errors.

4. The maximum likelihood orbit

Following Orlow (1939) we interject an application of formulae (25) and (26). Near opposition ($E \approx 0$) is obtained

$$(r-1)\cos\beta\dot{\lambda} + k = k\sqrt{\frac{1 + e\cos v}{r}}\cos i' \qquad (37)$$

$$(r-1)\dot{\beta} = k\sqrt{\frac{1 + e\cos v}{r}}\sin i' \qquad (38)$$

The square sum of which is

$$k^2(1 + e\cos v)/r = (r-1)^2c^2 + 2k(r-1)\cos\beta\dot{\lambda} + k^2 \qquad (39)$$

with

$$c^2 = \cos^2\beta\dot{\lambda}^2 + \dot{\beta}^2 \qquad (40)$$

Multiply (39) by r and subtract k^2

$$k^2 e\cos v = (r-1)(r(r-1)c^2 + 2kr\cos\beta\dot{\lambda} + k^2) \qquad (41)$$

The third order polynomial in r on the right has one or three real roots. The quadratic form has real roots if Tisserand's criterion for circular orbits is satisfied; in this case

$$k^2 e\cos v = c^2(r-1)(r-r_1)(r-r_2) \qquad (42)$$

where the two roots, r_1 and $r_2 = k^2/c^2r_1 \gg r_1$, correspond to pro– and retro–grade motions respectively. The maximum probability for the two unknowns $e\cos v$ and $e\sin v$ is by (19) obtained for $e = 0$.

Expanding a predicted place in these two small parameters gives the mean value and highest probability at the place predicted by a circular orbit.

For prograde motion and small eccentricity we obtain

$$r \simeq r_1 + k^2 e\cos v/c^2(r_1 - 1)(r_1 - r_2) \qquad (43)$$

If $\dot{\beta} \sim 0$

$$\cos\beta\dot{\lambda} = -c = -k/\sqrt{r_1}(\sqrt{r_1} + 1) \qquad (44)$$

After some reductions we obtain for $a \cong r + re\cos v$

$$a \simeq r_1 + \frac{r_1 e\cos v(r_1 - \sqrt{r_1} - 1)}{2(2r_1 - \sqrt{r_1} - 1)} \qquad (45)$$

For the particular value $r_1 = 2.62$ the coefficient to $e\cos v$ is exactly zero, and the semi–major axis **a** can be obtained from the daily motion. In this case the velocities of the planet and the Earth have the golden section ratio $\sqrt{r_1} = (\sqrt{5} + 1)/2$. This distance is remarkable also for the fact that the sidereal rotation period is equal to that of the apparent lightcurve (Kristensen, 1990).

The root–mean–square of the coefficient to $e\cos v$ in (45) over the asteroid belt is 0.119.By (19) the mean errors in the determination of **a** by (45) is ± 0.014.

The product of the four distribution functions in semi-major axis **a**, $e\cos v$, $e\sin v$, and $\sin i'$ gives a likelihood function in 4–dimensional space. The conditions (37) and (38) restrict the elements to a 2–dimensional surface on which the "maximum likelihood orbits" are defined as having large values of the product of probability functions.

Convenient parameters for the surface are r and $e\sin v$. The latter affects a and $\sin i'$ only in second order. The largest probability is thus obtained for $e\sin v = 0$ and the concept of "maximumn likelihood orbits" implies perihelion orbits.

Inside the circle

$$(\cos\beta\dot{\lambda}/k - \frac{1}{2})^2 + \left(|\dot{\beta}|/k - \frac{\sqrt{3}}{2}\right)^2 = 1 \qquad (46)$$

$e\cos v$ increases by (41) monotonously with r. Large values of r will thus be suppressed by the probability factors in both $e\cos v$ and $\sin i'$. The most likely orbits will in this case be in the inner belt.

For small motions in latitude, the maximum likelihood orbit is by (42) a circular orbit with $r = r_1$.

As an example where (41) has a positive minimum, and a circular orbit is not possible, we consider the case of 1938 HC given by Orlow (1939). Here $\dot{\lambda}\cos\beta = -0.227$ and $\dot{\beta} = -0.166$, and in the entire interval $1.65 \leq r \leq 3.95$ (41) gives $0.213 \leq e\cos v \leq 0.243$. The best guess for an orbit is obtained by the mean $a = 2.49 \pm 0.26$ computed with due regard to the reduced probabilities at large eccentricities and inclinations.

If an aphelion orbit ($e\cos v < 0$) is possible, then (41) shows that a circular orbit, with a lesser value of r, is also possible; and in general the circular orbit will be preferable.

5. Two–dimensional motion

The motion in plane coordinates $\boldsymbol{X} = (\cos\delta\Delta\alpha, \Delta\delta)$ is affected by two forces:

1) An acceleration towards the anti–Sun ($\boldsymbol{X}_\odot$) at distance E given by Lambert's theorem, $k^2\sin E(1 - r^{-3})/\Delta$. Near opposition $\Delta \simeq r - 1$ is approximately constant and the attraction is proportional to the distance to the anti–Sun. By this force the object moves in an ellipse, as if attached to the anti–Sun by an elastic spring.

2) The rate of change of the distance $\dot{\Delta}$ gives a foreshortening proportional to the velocities ($\dot{X}$, $\dot{Y}$) or a viscosity term.

The equations of motion in two–dimensions are then approximately

$$\ddot{X} = \frac{k^2}{\Delta}\left(1 - \frac{1}{r^3}\right)(X_\odot - X) - \frac{2\dot{\Delta}}{\Delta}\dot{X} \quad (47)$$

Near opposition we have $\dot{\Delta} \sim 0$ and to order t^3 the solution is

$$X(t) = X(0) + \dot{X}(0)t + \frac{k^2t^2}{2\Delta}\left(1 - \frac{1}{r^3}\right)\left(X_\odot(\tfrac{t}{3}) - X(\tfrac{t}{3})\right) \quad (48)$$

This formula may be used for about one week with errors of order $\pm 1'$, by using the mean value 0.56 ± 0.10 for the unknown coefficient $(1 - r^{-3})/\Delta$.

The equations (48) show that due to the rapid motion of the anti–Sun there will be a rather large term of order $(kt)^3/6\Delta$ in longitude.

A much more slow and smooth motion in two dimensions is obtained by a transformation of (α, δ) to the new variables (T, I) introduced by Kristensen(1990). A circular orbit with $a = 2.74$ and an argument of latitude 90° are assumed. The latter determines the minimum inclination I. The longitude of the planet and Earth, in their respective orbits, are equal at time T. Already constant values of the new variables give a qualitative correct description of the opposition loops and accounts for the main effects caused by the rapid motion of the Earth. A polynomial approximation in these variables will therefore give a good approximation, especially if terms of third order are included to describe the entire opposition loop with its two stationary points.

6. Linear extrapolation

During the first few days the motion will necessarily be described by a linear function of time, as the second and higher order terms will not be significantly larger than the errors of observation $\pm\sigma$.

Linear extrapolation in a coordinate, say α, from observations at time t_0 and t_1 to t_2 has two types of errors:

The measuring errors give the following error on the predicted place

$$\pm\sigma \cdot \sqrt{2} \cdot \sqrt{\left(\frac{t_2 - (t_0 + t_1)/2}{t_1 - t_0}\right)^2 + \frac{1}{4}} \quad (49)$$

The second error is due to the neglected second order term

$$\frac{1}{2}\frac{d^2\alpha}{dt^2}(t_2 - t_0)(t_2 - t_1) \quad (50)$$

This remainder term could be smaller by interpolation to t_1 from t_0 and t_2:

$$\frac{1}{2}\frac{d^2\alpha}{dt^2}(t_1 - t_0)(t_1 - t_2) \quad (51)$$

For the present purpose nothing could be gained, however, because the density of points to be compared with the position at t_1 will be increased by a factor $(t_2 - t_0)/(t_1 - t_2)$, which will make the critical parameter p the same as before.

A numerical example with $\sigma = \pm 1''$, $\mu = 20$ and the critical parameter $p = 0.01$ may show that the measuring errors may give a limitation only for positions referring to the same night

$$p = 2\pi \cdot 20 \cdot 3600^{-2} \cdot 2 \cdot \left(\frac{t_2 - (t_0 + t_1)/2}{t_1 - t_0}\right)^2 \le 0.01 \quad (52)$$

or

$$\mid t_2 - (t_0 + t_1)/2 \mid \le 23 \mid t_1 - t_0 \mid \quad (53)$$

The acceleration of (α, δ) towards the anti-Sun may be estimated by (48)

$$\frac{d^2\alpha}{dt^2} \simeq (0.56 \pm 0.10)k^2 \sin E = 0.0094 \sin E^\circ/d^2 \quad (54)$$

If the elongation E is smaller than 30° is obtained

$$p \simeq 2\pi \cdot 20 \cdot (0.0047/2)^2(t_2 - t_0)^2(t_2 - t_1)^2 \le 0.01 \quad (55)$$

or

$$(t_2 - (t_0 + t_1)/2)^2 \le 3.80 + (t_1 - t_0)^2/4 \quad (56)$$

This shows that planets may be followed by linear extrapolation in (α, δ) for about two days.

The smooth coordinates T and I are much more suited for extrapolation. Table 2 in Kristensen (1990) shows that the second order difference in 10-days intervals have order of magnitude $\Delta^2 T \simeq 0^d\!.05$ and $\Delta^2 I \simeq 0^\circ\!.02$. In T, 1^d corresponds to $1^\circ\!.211$ in longitude and in I, 1° corresponds to $2.74/1.74 = 1^\circ\!.57$ in latitude. The second derivatives are of order $^d\!05 \cdot 1^\circ\!.211/100 = 0.00061$ and $0^\circ\!.02 \cdot 1.57/100 = 0.00031$.

$$p \simeq 2\pi \cdot 20 \cdot \left((t_2 - t_0)(t_2 - t_1)/2\right)^2 \cdot 0.00061 \cdot 0.00031 \simeq .01 \quad (57)$$

or

$$\left(t_2 - (t_0 + t_1)/2\right)^2 \simeq 41 + (t_1 - t_0)^2/4 \quad (58)$$

This means the motion can be followed by linear extrapolation in the coordinates (T, I), within a week before and after a central pair of positions.

Table 2. Five of the eight ephemeris positions from EMP 1989 for (368) Heidea are transformed to the slowly varying time of opposition T and minimum inclination I. The linear combinations $L = (10f_0 - 28f_1 + 21f_2)/3$ and $R = (7f_6 - 4f_7)/3$ based on positions in the first and second lunation respectively should be equal apart from a remainder term of order $(kt)^4$. The difference $L - R$ corresponds to errors $-3'$ and $+0'\!.7$ in longitude and latitude. These errors are small enough to identify R and L across the 40 days gap between the last and first observations of the two independent groups. In surveys positions can be computed independently from the first and second lunation and plotted and identified on the same graph.

f_i	T	I	$3w$
f_0	$10^d\!.136$	$-5^\circ\!.476$	+10
f_1	10.016	-5.317	−28
f_2	10.031	-5.170	+21
f_6	10.578	-4.700	+7
f_7	10.589	-4.606	−4
L	10.521	-4.818	
R	10.563	-4.825	
$L - R$	$-0^d\!.041$	$+0^\circ\!.007$	

Table 3. The discovery positions of 1991 VS_5 and 1991 VU_5 transformed to the smooth variables T and I. The weights w gives by linear extrapolation the position L on November 9.891 from the first three dates and the same position R from the last two dates. The weights w are determined to minimize the observational errors and the second order (t^2) terms in the difference $L-R$. These differences become small enough for a reliable identification of the November 12 trails without a computation of uncertain orbits. The method uses *all* information, including ends of trails, and the same weights w for many objects.

1991 Nov.	T	I	T	I	w
2.21389	16ᵈ06584	−5°84760	16ᵈ29878	−6°89246	20.475
2.24028	16.06428	−5.84744	16.29754	−6.89413	−21.580
6.13125	15.99228	−5.83981	16.24266	−7.11737	+2.105
12.14792	15.83514	−5.81343	16.14316	−7.44380	86.527
12.17431	15.83389	−5.81315	16.14204	−7.44523	−85.527
L	15.94466	−5.83465	16.20741	−7.32986	
R	15.94205	−5.83738	16.23895	−7.32150	
$L-R$	+0′2	+0′3	−2′3	−0′8	

7. Linking two lunations

To demonstrate how two consecutive lunations can be linked by interpolation methods without orbit computation, we consider the example of (368) Heidea in Table 2. The first three and the last two standard positions from the Ephemeris of Minor Planets 1989 are transformed to the smooth variables T and I.

The seventh positions (f_6) may be obtained from the first three (f_0, f_1, f_2) and the last (f_7) by Lagrange's interpolation formula. A rearrangement and multiplication by a constant, in order to make the sum of the coefficients equal to 1, give:

$$(10f_0 - 28f_1 + 21f_2)/3 = (7f_6 - 4f_7)/3 \tag{59}$$

The left side (L) of this equation is derived only from the positions in the first and the right side (R) only from positions in the following lunation. Considering the 40 days gap between f_2 and f_6, the difference $L-R$ is small. A sample gives for the root–mean–square $\sigma_T = \pm^{\mathrm{d}}0558$ and $\sigma_I = \pm^{\circ}0083$ including the rounding of the ephemeris. This corresponds to mean errors in longitude and latitude

$$\sigma_\lambda = 0.0558 \cdot 1.211 = 0^{\circ}068,\ \sigma_\beta = 0.0083 \cdot 1.57 = 0^{\circ}013 \tag{60}$$

For the identification is assumed that the objects are linked *within* but not *between* the lunations. In photographic surveys the observing times are equal for all objects, and the fictitious positions R and L can be computed independently and plotted on the same graph, where nearby positions may be identified.

The two positions R and L may be thought of as positions at time $t = 4.67$, derived from the first and second groups of observations. This is correct only as far as the linear terms are concerned and will preserve the density (μ) of the objects. However, strictly speaking, R and L are not points on the trajectory, because the t^2 terms are computed for $t = 4.32$ and the t^3 terms for $t = 3.60$.

The errors (60) are small compared with predictions by perihelion orbits. The critical parameter p is:

$$p = 2\pi \cdot 20 \cdot 0.068 \cdot 0.013 = 0.11 \tag{61}$$

In practice the time interval would be about half that of this example. The errors are of order t^4 and will be decreased by a factor 2^{-4} in each coordinate. The errors can not be reduced below short period terms and parallax.

The general formula is completely symmetric in the time arguments t_i; it is based on the smallness of order k^n of the n^{th} divided difference

$$[t_0, t_1, \cdots t_n] = \sum_{k=0}^{n} \frac{f_k}{\pi_n'(t_k)} = \frac{f^{(n)}(\xi)}{n!} \approx 0 \tag{62}$$

where π_n' is the derivative of

$$\pi_n(t) = (t-t_0)(t-t_1)\cdots(t-t_n) \quad \text{and} \quad t_0 \le \xi \le t_n. \tag{63}$$

There is a great variety of formulae of which (62) is only an example. The order of the polynomial approximation may not be increased by additional positions. Instead the square sum of the coefficients could be minimized in order to reduce the effects of short period terms.

As a final example we consider some plates by E.W. Elst (1992) given in Table 3. The positions for the new objects 1991 VS_5 and 1991 VU_5 are transformed to the smooth parameters T and I. A weighted average of the first three values gives L and of the last two gives R. The weights w represent linear motion exactly. However, the difference $L-R$ is affected by two types of errors: the individual observational errors, which we write $0 \pm 1''$, and the second order (t^2) terms with an estimated coefficient $0 \pm 4'.4$. The weights w are determined to minimize the square sum of these errors. The difference $L-R$ of order $\pm 2'$ is small enough to identify the November 12 trails unequivocally. The method can handle many objects with the same set of weights, and has the advantage that all information, including the slope of trails, may be utilized. The large coefficients, caused by the short exposures, should be reduced in case the mean distance between the trails is not larger than the error above of order $\pm 2'$.

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